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Dual Non-Abelian Duality and the Drinfeld Double

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Abstract

The standard notion of the non-Abelian duality in string theory is generalized to the class of σ-models admitting 'non-commutative conserved charges'. Such σ-models can be associated with every Lie bialgebra $(\mathcal{G},\tilde{\mathcal{G}})$ and they possess an isometry group iff the commutant $[\tilde{\mathcal{G}},\tilde{\mathcal{G}}]$ is not equal to $\tilde{\mathcal{G}}$. Within the enlarged class of the backgrounds the non-Abelian duality *is* a duality transformation in the proper sense of the word. It exchanges the roles of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ and it can be interpreted as a symplectomorphism of the phase spaces of the mutually dual theories. We give explicit formulas for the non-Abelian duality transformation for any $(\mathcal{G},\tilde{\mathcal{G}})$. The non-Abelian analogue of the Abelian modular space $O(d,d;\mathbf{Z})$ consists of all maximally isotropic decompositions of the corresponding Drinfeld double.

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1 Introduction

Duality symmetry plays an important role in string theory as a tool for disentangling its full symmetry structure. It deepens our understanding of the geometry of the spacetime from the string point of view, because it relates apparently different, but still equivalent, backgrounds in the σ-model approach. The Abelian target space duality kept receiving the attention of string theorists in the past few years and much progress has been achieved in revealing the consequences of the symmetry in string theory and in classifying backgrounds related by the Abelian duality group $O(d,d;\mathbf{Z})$ [1, 2, 3, 4, 5, 6, 7]. From the σ-model point of view the necessary condition to work out a dual to some background was that the latter possess an Abelian group of isometries. Thus, the class of string backgrounds concerned was rather restricted and many physically relevant classical string vacua were excluded from the consideration.

An important materialization of the suspicion that the duality symmetries should be associated also with the non-Abelian isometries of the target manifold was achieved by de la Ossa and Quevedo [8]. They gauged the non-Abelian isometries of σ-models and constrained the field strenth F to vanish. The dual action was then obtained by integrating out the gauge fields and the Lagrange multipliers had become coordinates of the dual manifold. In the series of subsequent investigations [9, 10, 11, 12, 13, 14, 15, 16, 17] other relevant insights were obtained, still the notion of 'non-Abelian duality' was lacking some of the key features of its Abelian counterpart. The non-Abelian isometry group of the dual space was always smaller; also, a canonical procedure was missing that would yield the original theory if one is given only its non-Abelian dual. For the above reasons it was somewhat considered as a misnomer to call the non-Abelian duality a duality transformation [5].

In this contribution we attempt to cure those drawbacks of the non-Abelian duality. The starting point of our considerations was the fact that a non-Abelian dual, even without isometries, is still equivalent to an apparently different σ-model. This made us believe that the relevant algebraic structure for the existence of a non-Abelian T-duality is not necessarily the existence of the group of the isometries of the background, but some other structure that shows up only in special cases as an isometry group. We even show that what is called the non-Abelian duality in the present nomenclature, could also be referred to as a sort of semi-Abelian duality. Indeed, as a result

of our analysis, we are able to present a lot of examples of the full non-Abelian duality, when the duality is performed in both directions without the isometry.

In looking for an explicit criterion, when a given σ-model is the non-Abelian dual of another one, the clear geometric understanding of the Abelian duality, which we presented in our previous work [18], proved very useful. In the formalism developed there the central role was played by the Noetherian forms on the world-sheet associated with the Abelian isometry group G_0 of the target. Those forms were closed (hence integrable) on any extremal string surface by virtue of the symmetry and their integrals turned out to be the coordinates of the dual target. Both the original M and the dual target $\tilde{M}$ were embedded into one manifold M_E which turned out to have a natural structure of a fibre bundle over M/G_0 (or $\tilde{M}/\tilde{G}_0$) with the fibre $G_0 \times \tilde{G}_0$. The canonical symplectic structure on $G_0 \times \tilde{G}_0$ gave the difference between the original and the dual action, or, in other words, Buscher's formula [1].

All relevant steps of the previous construction can be repeated for a much general class of targets, however. We again need the action of a (possibly non-Abelian) group G on the σ-model target which gives rise to the Noetherian forms. If G does not act as an isometry, the forms are not closed even on the extremal surfaces. They can still be integrable, however! Imagine that we organize the Noetherian forms α_a $(a = 1, 2, ..., dim G)$ into a Lie algebra valued form $\alpha = \alpha_a \tilde{T}^a$, where $\tilde{T}^a$ are the generators of some (dual) Lie algebra $\tilde{\mathcal{G}}$. Suppose that the target has the property that on the extremal string surfaces the form α is a flat connection, i.e. it satisfies the Maurer-Cartan equation

$$d\alpha_a - \frac{1}{2}\tilde{c}_a{}^{kl}\alpha_k \wedge \alpha_l = 0, \tag{1}$$

where $\tilde{c}^{kl}_a$ are the structure constants of $\tilde{\mathcal{G}}$. Then the form α is integrable, which means that there exists a map $\tilde{g}(\tau, \sigma)$ from the worldsheet to the dual group $\tilde{G}$ such that

$$\alpha = d\tilde{g}\ \tilde{g}^{-1}. \tag{2}$$

We call this property a *non-commutative conservation law.* If the group G acts freely on the original target then we can choose a preferred system of coordinates (y, g) where y's label the orbits of G in the target M and $g \in G$. We shall see that if the connection α is flat, extremal strings live naturally in an extended manifold M_E having the structure of a fibre bundle over M/G

(or $\tilde{M}/\tilde{G}$) with the fibre being the Drinfeld double $(G, \tilde{G})$. The canonical symplectic structure on the double gives the difference between the original and the dual action and, hence, the non-Abelian Buscher's formula. The group $\tilde{G}$ acts naturally on the dual target $\tilde{M}$ and the dual Noetherian form can be organized in the $\mathcal{G}$-valued form $\tilde{\alpha} = \tilde{\alpha}_a T^a$. So the procedure can be repeated, returning to the original target.

The analogue of the Abelian modular space $O(d, d, \mathbf{Z})$ is given by the structure of the Drinfeld double, namely by the classification of the decompositions of the algebra of the double in the pairs of maximally isotropic subalgebras with respect to the *ad*-invariant bilinear form on the double. Such decompositions can be constructed by means of the automorphisms of the Drinfeld double, which naturally form a subgroup of $O(dim\mathcal{G}, dim\tilde{\mathcal{G}}, \mathbf{Z})$. However, unlike in the Abelian case, the automorphisms do not necessarily exhaust all possibilities.

The standard non-Abelian duality of de la Ossa and Quevedo [8] is the special case of our treatment. The dual group $\tilde{G}$ is Abelian and the corresponding Drinfeld double is the cotangent bundle of the group manifold G with its canonical symplectic form. The Abelian duality is described by the double where both groups are Abelian. It has the topology of a $2dimG$-dimensional torus and its group of automorphisms (preserving the invariant bilinear form) is $O(dimG, dim\tilde{G}, \mathbf{Z})$.

In section 2 of our note we give the explicit criterion when a given σ-model is the non-Abelian dual of another and a straightforward prescription of how to reconstruct the original model from its dual (or how to perform the non-Abelian duality in both directions). We emphasize that no relevant local algebraic structure is lost upon performing the duality transformation. We describe in detail the geometric structure of the non-Abelian duality by means of the lift of the dynamical characteristics of string to the Drinfeld double.

In section 3 we shall discuss the interpretation of the non-Abelian duality in terms of a canonical transformation. In section 4 we give explicit formulas for the non-Abelian duality transformations for a generic Lie bialgebra and discuss their projective character.

In the concluding section 5 we describe the relevant structure of the Drinfeld double which gives rise to the non-Abelian analogue of the Abelian modular space $O(d, d, \mathbf{Z})$; we discuss the dressing transformations and touch the issue of integrability. We finish with comments about quantization, in par-

ticular about the conformal invariance, the dilaton and possible emergence of the quantum group structure.

2 Non-Abelian duality and Lie bialgebras

In what follows we shall consider two-dimensional σ-models described by a metric G_{ij} on the target manifold M and a globally defined two-form B_{ij} on M with the action

$$S = \int dz d\bar{z}(G_{ij}(x) + B_{ij}(x))\partial x^i \bar{\partial} x^j \equiv \int dz d\bar{z} E_{ij} \partial x^i \bar{\partial} x^j. \tag{3}$$

Suppose that a group G acts freely on M. We can associate to this action the Noetherian forms on the world-sheet given by

$$J_a = v_a^i(x) E_{ij} \bar{\partial} x^j d\bar{z} - v_a^i(x) E_{ji} \partial x^j dz, \tag{4}$$

where $v_a^i(x)$ are the (left-invariant) vector fields corresponding to the right action of G on M. They can be defined also when G is not the isometry of the target, by varying the action with respect to the G transformations with the world-sheet dependent parameters $\varepsilon^a(z, \bar{z})$, i.e.

$$\delta S = S(x + \varepsilon^a v_a) - S(x) = \int \varepsilon^a \mathcal{L}_{v_a}(L) + \int d\varepsilon^a \wedge J_a. \tag{5}$$

If the Lie derivative of the Lagrangian $\mathcal{L}_{v_a}(L)$ vanishes, then the forms J_a are closed on the extremal surfaces $x^i(z, \bar{z})$. We shall look for a condition on E_{ij} which would guarantee that the forms J_a on the extremal surfaces satisfy

$$dJ_a = \frac{1}{2} \tilde{c}_a^{\ kl} J_k \wedge J_l. \tag{6}$$

Here $\tilde{c}_a^{\ kl}$ are the structure constants of some Lie algebra $\tilde{\mathcal{G}}$. From Eq. (3) it follows that

$$\mathcal{L}_{v_a}(L) = \frac{1}{2} \tilde{c}_a^{\ kl} J_k \wedge J_l \tag{7}$$

or, in other words,

$$\mathcal{L}_{v_a}(E_{ji}) = \tilde{c}_a^{\ kl} v_k^m v_l^n E_{mi} E_{jn}. \tag{8}$$

If the condition (8) holds we may associate to each extremal surface $x^i(z,\bar{z})$ a mapping $\tilde{g}(z,\bar{z})$ from the world sheet into the dual group $\tilde{G}$ such that

$$J_a = d\tilde{g}\ \tilde{g}^{-1} \tag{9}$$

or

$$\tilde{g} = P \exp \int_\gamma J_a \tilde{T}^a, \tag{10}$$

where P means the path-ordered exponential. We shall refer to the Lagrangians fulfilling (8) as to the σ-models admitting non-commutative conservation laws.

Note from (10) that $\tilde{g}$ is defined up to the homotopy class of the curve γ. If, for instance, we integrate around a closed non-contractible loop on the world sheet of the closed string, the integral (10) gives a fixed element of the dual group $\tilde{G}$, which we refer to as a 'charge'. However, this charge is not a number but a non-commutative object. If we run around the loop twice we have to multiply charges rather than add them.

Condition (8) in fact requires that a certain compatibility requirement should be imposed on the structure constants of the original and dual Lie algebras. This requirement is the integrability condition of the set of the first-order differential equations (8). It is easy to see that the condition reads

$$\tilde{c}_k^{\ ac} c^l_{\ fa} - \tilde{c}_k^{\ al} c^c_{\ fa} - \tilde{c}_f^{\ ac} c^l_{\ ka} + \tilde{c}_f^{\ al} c^c_{\ ka} - \tilde{c}_a^{\ lc} c^a_{\ fk} = 0. \tag{11}$$

Amazingly, this is the standard relation which must be obeyed by the structure constants of the Lie bialgebra $(\mathcal{G},\tilde{\mathcal{G}})$ [19, 20, 21]! This condition is manifestly dual, hence we expect that there exists an equivalent dual σ-model where the roles of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are exchanged. Obviously, the dual model $\tilde{E}_{ij}$ should fulfil

$$\mathcal{L}_{\tilde{v}_a}(\tilde{E}_{ij}) = c_a^{\ kl} \tilde{v}^m_k \tilde{v}^n_l \tilde{E}_{mi} \tilde{E}_{jn}. \tag{12}$$

For the sake of clarity we shall first discuss the case in which the group G acts on the target transitively (and freely), i.e. the target itself can be identified with the group manifold. Then there is a very easy and beautiful way of solving Eqs. (8) and (12), using the concept of the Drinfeld double D [19, 20, 21]. The latter is the connected group corresponding to the Lie algebra double $\mathcal{D}$ and containing both groups G and $\tilde{G}$. The double $\mathcal{D}$ is

equal to $\mathcal{G}+\mathcal{G}^*$ as the vector space with the Lie bracket[2]

$$[X+v, Y+w] \equiv [X,Y] + [v,w]^* - ad^*_X w + ad^*_Y v + ad^*_w X - ad^*_v Y. \quad (13)$$

Here ad^*_X is the usual ad^*-operator for the Lie algebra $\mathcal{G}$ acting on $\mathcal{G}^*$. The symbol ad^*_w corresponds to the coadjoint action of the Lie algebra $\mathcal{G}^*$ on its dual space $\mathcal{G}$. Note that both groups G and $\tilde{G}$ are embedded into D. The algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ form the maximally isotropic subspaces of $\mathcal{D}$ with respect to the ad-invariant non-degenerate bilinear form

$$(X+v, Y+w) \equiv \langle X, w\rangle + \langle Y, v\rangle. \quad (14)$$

The symbol $\langle .,. \rangle$ means the canonical pairing between the algebra $\mathcal{G}$ and its dual $\mathcal{G}^*$.

Consider the tangent space $T_eD \cong \mathcal{D}$ at the unit element $e \in \mathcal{D}$. (Of course, e is the unit of both G and $\tilde{G}$ at the same time.) In T_eD we can take a d-dimensional subspace $\mathcal{E}$ which is the graph[3] of a non-degenerate linear mapping $E : \mathcal{G} \to \tilde{\mathcal{G}}$. The subspace $\mathcal{E} \subset T_eD$ can be transferred to every point $g \in G(\hookrightarrow D)$ by the right action of G itself. At the point $g \in G$ $T_gD \cong \mathcal{D}$ again and its decomposition into $\mathcal{G}+\tilde{\mathcal{G}}$ is given by the *left* action of g on T_eD. Hence, we have defined at every $g \in G$ a non-degenerate linear mapping $E_g : \mathcal{G} \to \tilde{\mathcal{G}}$ with the graph $\mathcal{E}_g$. Since $\tilde{\mathcal{G}}$ is canonically identified with $\mathcal{G}^*$, we have obtained a matrix $E_{ab}(g)$. It is straightforward to check that $E_{ab}(g)$ solves Eq. (8)[4]. Obviously, the solution of Eq. (12) can be obtained in the same way by transferring a subspace $\tilde{\mathcal{E}} \subset T_eD$ into the whole $\tilde{G}$ by the right action of $\tilde{G}$ itself. It is natural (and also supported by the Abelian duality case [18]) to conjecture that the mutually dual σ-models are obtained by taking $\tilde{\mathcal{E}} = \mathcal{E}$. In other words, we transfer the same subspace $\mathcal{E} \subset T_eD$ onto G and $\tilde{G}$.

Now we have to prove that the σ-models E_g and $\tilde{E}_{\tilde{g}}$ are equivalent. First of all we map every solution $g(z,\bar{z})$ of the original model into a solution $\tilde{h}(z,\bar{z})$ of the dual one. Following Eq. (9), $g(z,\bar{z})$ can be considered as a surface

$$f(z,\bar{z}) = g(z,\bar{z})\tilde{g}(z,\bar{z}) \quad (15)$$

[2] $\tilde{\mathcal{G}}$ is identified with the dual space $\mathcal{G}^*$ of $\mathcal{G}$.

[3] By the graph we mean the set $\{t \in \mathcal{G}, t + E(t,.)\} \subset \mathcal{G}+\tilde{\mathcal{G}}$.

[4] In fact, it is the general solution, $\mathcal{E}$ playing the role of the initial value for the first-order equation (8).

in the Drinfeld double D, where the multiplication is taken in D. It is known [20] that the following two decompositions are applicable for every $f \in D$:

$$f(z,\bar{z}) = g(z,\bar{z})\tilde{g}(z,\bar{z}) = \tilde{h}(z,\bar{z})h(z,\bar{z}). \tag{16}$$

We show that $\tilde{h}(z,\bar{z}) \in \tilde{G}$ defined by Eq. (16) is indeed a solution of the dual model $\tilde{E}_{\tilde{g}}$ and $h(z,\bar{z})$ is associated to it by the dual analogue of Eq. (9). The easiest way to show this consists in finding a condition when a surface $l(z,\bar{z})$ in D can be obtained by lifting an extremal solution of the model E_g via (15). The conditions read

$$\bar{\partial} l\ l^{-1} \in \mathcal{E}, \qquad \partial l\ l^{-1} \in \mathcal{E}^{\perp}, \tag{17}$$

where the orthogonal complement is taken with respect to the bilinear form (14). Before proving the statement, note that the conditions (17) do the right job because they do not depend in any way on the choice of the group G or $\tilde{G}$ in the double D. Thus the existence of the extremal $\tilde{h}(z,\bar{z})$ with the associated $h(z,\bar{z})$ is obvious, since only the extremal surfaces are liftable. The choice $\tilde{\mathcal{E}} = \mathcal{E}$ is crucial for the statement, of course.

The proof that the conditions of 'liftability' are given by the relations (17) requires a little geometry. Suppose that an element $g \in G(\hookrightarrow D)$ lies on the surface $l(z,\bar{z})$ in D. A vector $\partial g = \varepsilon^a v_a$ at g on an extremal surface in G is lifted into $T_g D$ via Eq. (9), i.e. it becomes

$$\partial f = \varepsilon^a v_a + \varepsilon^a J_a = \varepsilon^a v_a - E(., \varepsilon^a v_a). \tag{18}$$

The last equality follows from the definition (4) of the currents. In a similar way

$$\bar{\partial} f = \bar{\partial} g + E(\bar{\partial} g, .). \tag{19}$$

Clearly, if $l(z,\bar{z})$ is the liftable surface, ∂l and $\bar{\partial} l$ have to obey Eqs. (18) and (19). In other words

$$\bar{\partial} l \in \mathcal{E}_l, \qquad \partial l \in \mathcal{E}_l^{\perp}. \tag{20}$$

Because $\mathcal{E}_l$ was obtained from $\mathcal{E}$ by the right action of l, the conditions (17) follow. If we are at a point $l(z,\bar{z})$ which does not lie at G we may transfer it there by the right action of $\tilde{G}$, because the lift of the extremal surface in G via Eq. (9) is defined up to the right action of $\tilde{G}$.

3 Non-Abelian duality as a symplectomorphism

So far we have constructed the mapping between the phase spaces of the original and the dual σ-models. This mapping has a constructive character because it guarantees that solving the original σ-model we can also solve the apparently different σ-model. In this sense the two theories are equivalent. We shall show, however, that the equivalence of the models can be understood in a much stronger way, namely the mapping between the phase spaces preserves their natural symplectic structure [5]. To demonstrate this we have to extend our so far local analysis by the discussion of the boundary conditions. According to the comments after Eq. (10), the integration of the original Maurer-Cartan form (4) around the non-contractible loop on the world sheet of the closed string gives the non-commutative charge belonging to the dual group. We shall restrict the phase spaces of the models to the configurations having the unit charge, otherwise the lifting of the string into the Drinfeld double does not give a closed loop. Such a restriction renders degenerate the natural symplectic form Ω_{Ph} coming from the action. To cure the problem we have to perform a generalized Marsden-Weinstein reduction. As already mentioned, the lift $g(z,\bar{z})\tilde{g}(z,\bar{z})$ of an extremal surface into the double is defined up to the right multiplication by a constant element $\tilde{g}_0 \in \tilde{G}$. All such lifts we identify in the dual phase space and we proceed similarly in the dual case. Only applying this procedure does the mapping between (the reduced) phase spaces become one-to-one and, moreover, the restricted form Ω_{Ph} becomes non-degenerate. Now we prove that the non-Abelian duality is a symplectomorphism of the (reduced) phase spaces.

Let LG be the loop space of the target G. As usual, we obtain the phase space from the cotangent bundle T^*LG, on which there is the canonical symplectic form $\Omega_{Ph} = d\theta_{Ph}$. Namely, we identify some submanifold in T^*LG and then factorize it appropriately[6]. The construction goes as follows: if we have a string world-sheet F and a loop l on it, then we define a corresponding element $l_F \in T^*_l LG$. To describe how l_F acts on a vector $u \in T_l LG$, first

[5] For the special case of the non-Abelian duality between the $SU(2)$ group and its coalgebra the statement has been proved in [22].

[6] We proceed conceptually as in the case of a relativistic particle in a background; in the σ-model case the submanifold is defined by the Virasoro constraints.

realize that u can be thought of as a family of vectors $u(X) \in T_X G$ where X runs along l. Then

$$l_F(u) \equiv \oint_l J_a u^a(X). \tag{21}$$

If we take all l_F's for all possible F's we obtain the mentioned submanifold of T^*LG. Now we identify all l_F's coming from the same extremal F and obtain the (unreduced) phase space.

Let H be a surface (i.e. a 2-parametric family of on-shell strings) in the unreduced phase space of the original model such that all strings have the unit charge. Let there also be a loop on each $F \in H$ $l(F)$. Then by (21)

$$\int_H \Omega_{Ph} = \oint_{\partial H} \theta_{Ph} = \oint_{\bigcup_{F\in\partial H} l(F)} J \tag{22}$$

Here J is understood as a two-form on the two-dimensional closed surface $\bigcup_{F\in\partial H} l(F)$ in the target G. The form J_a on the world-sheet is to be saturated by the vectors on the surface tangent to the loops $l(F)$ and the index a is saturated by the vectors connecting the infinitesimally close loops $l(F)$. Now we lift the family H into a family H_D of surfaces in D; H_D is defined up to an independent shift by the right action by a constant element from $\tilde{G}$ on each surface in H_D. We project the family H_D into $\tilde{G}$ according Eq. (16), thus obtaining the family $\tilde{H}$ of extremal surfaces in $\tilde{G}$. We have to prove that

$$\oint_{\bigcup_{F\in\partial H} l(F)} J = \oint_{\bigcup_{\tilde{F}\in\partial\tilde{H}} l(\tilde{F})} \tilde{J}. \tag{23}$$

We stress that this relation holds in spite of the ambiguity in the definition of $\tilde{H}$. This means that the symplectic form $\tilde{\Omega}_{Ph}$ is well defined on the *reduced* dual phase space. The dual statement holds, too.

We shall compare the two expressions in Eq. (23), using the common lifted family H_D. We demonstrate that if t_D and u_D are vectors at a point P of a lifted surface F^D, t_D tangent to F^D and u_D arbitrary, then

$$\tilde{J}_a(\tilde{t})\tilde{u}^a - J_a(t)u^a = \Omega_D(t_D \wedge u_D), \tag{24}$$

where $\tilde{t}, \tilde{u} \in T\tilde{G}$; $t, u \in TG$ are given by the projections (inverse lifts) and Ω_D is the canonical symplectic form on D [20], to be written explicitly in what follows. Consider the subspaces $S_{R(L)}$ and $\tilde{S}_{R(L)}$ obtained by the right (left)

action of P on $\mathcal{G}+\tilde{\mathcal{G}}$ embedded in T_eD. We now define a linear mapping $\Pi_{R\tilde{R}}$ in T_PD as the projection on $\tilde{S}_R$ with the kernel S_R and accordingly for the other combinations of the indices. By definition, $\tilde{J}_a(\tilde{t})\tilde{u}^a = (t_D, \Pi_{L\tilde{R}} u_D)$ and $J_a(t)u^a = (t_D, \Pi_{\tilde{L}R} u_D)$, where the round bracket is the invariant bilinear form (14) in the double. According to Ref. [20][7]

$$\Omega_D(t_D, u_D) = (t_D, (\Pi_{R\tilde{R}} - \Pi_{\tilde{L}L})^{-1} u_D). \tag{25}$$

Thus we have to prove that

$$(\Pi_{R\tilde{R}} - \Pi_{\tilde{L}L})(\Pi_{L\tilde{R}} - \Pi_{\tilde{L}R}) = 1. \tag{26}$$

We do it easily

$$\begin{gathered}(\Pi_{R\tilde{R}} - \Pi_{\tilde{L}L})(\Pi_{L\tilde{R}} - \Pi_{\tilde{L}R}) = \\ = \Pi_{R\tilde{R}}\Pi_{L\tilde{R}} - \Pi_{\tilde{L}L}\Pi_{L\tilde{R}} + \Pi_{\tilde{L}L}\Pi_{\tilde{L}R} = \Pi_{L\tilde{R}} - \Pi_{\tilde{L}L}\Pi_{L\tilde{R}} + \Pi_{\tilde{L}L} = \\ = (\Pi_{L\tilde{R}} + \Pi_{\tilde{R}L}) + (\Pi_{\tilde{L}L} - \Pi_{\tilde{R}L} - \Pi_{\tilde{L}L}\Pi_{L\tilde{R}}) = 1 + 0.\end{gathered} \tag{27}$$

Now from Eq. (24) we can conclude

$$\oint\limits_{\bigcup_{\tilde{F}\in\partial\tilde{H}} l(\tilde{F})} \tilde{J} - \oint\limits_{\bigcup_{F\in\partial H} l(F)} J = \oint\limits_{\bigcup_{F^*\in\partial H^*} l(F^*)} \Omega_D = 0 \tag{28}$$

because Ω_D is closed and the closed surface over which we integrate is a boundary.

4 Projective transformations

In this section we give explicit formulas for the non-abelian duality transformations for a generic Lie bialgebra $(\mathcal{G}, \tilde{\mathcal{G}})$. They easily follow from the general discussion in section 2. When the group G acts transitively on the target, the explicit formula for the σ-model E_g is given by

$$E_g^t = d(g)E_0^t(a(g) + b(g)E_0^t)^{-1}, \tag{29}$$

[7]We are much indebted to B. Jurčo for pointing out to us the existence of the symplectic structure Ω_D, which we badly needed.

where t means the transposition of matrices and the functions a, b, d are the components of the adjoint action of g on $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}}$. In other words

$$g^{-1} \begin{pmatrix} X \\ v \end{pmatrix} g \equiv \begin{pmatrix} a(g) & b(g) \\ 0 & d(g) \end{pmatrix} \begin{pmatrix} X \\ v \end{pmatrix}, \tag{30}$$

where $X \in \mathcal{G}$ and $v \in \tilde{\mathcal{G}}$. The dual σ-model is obtained by

$$\tilde{E}^t_{\tilde{g}} = \tilde{d}(\tilde{g}) E_0^{t\,-1} (\tilde{a}(\tilde{g}) + \tilde{b}(\tilde{g}) E_0^{t\,-1})^{-1}. \tag{31}$$

Note that the constant matrix E^t_0 in (29) was replaced by its inverse in (31) because the subspace $\mathcal{E}$ in $T_e D$ is the graph of the inverse of E^t_0 from the dual point of view.

We may illustrate the content of formulas (29) and (31) for the case of the non-abelian duality of de la Ossa and Quevedo [8]. The double D is simply the cotangent bundle T^*G with the structure of the semi-direct product of G and the abelian group $\mathbf{R}^{dim G}$. For simplicity we take $E_0 = Id$. Then Eq. (29) gives

$$E_g = Id \tag{32}$$

and Eq. (31)

$$(\tilde{E}^{-1}_\chi)^{ab} = \delta^{ab} + \chi^k c^{ab}_k, \tag{33}$$

where χ^k are coordinates on the fibre. The corresponding Lagrangians are respectively

$$L = Tr(g^{-1} \partial g g^{-1} \bar{\partial} g) \tag{34}$$

and

$$\tilde{L} = \tilde{E}_{ab}(\chi) \partial \chi^a \bar{\partial} \chi^b. \tag{35}$$

We now present the analogues of the Buscher formula for the abelian duality. We consider the case in which G does not act transitively. The coordinates labelling the orbits of G in the target M, we denote $y^\alpha (\alpha = 1, \ldots, n)$. The matrix of the σ-model E_{ij} has both types of indices corresponding to y^α and g. The Lagrangian reads

$$\begin{aligned} L = E_{\alpha\beta}(y) \partial y^\alpha \bar{\partial} y^\beta + E_{\alpha b}(y, g) \partial y^\alpha (g^{-1} \bar{\partial} g)^b + \\ + E_{a\beta}(y, g)(g^{-1} \partial g)^a \bar{\partial} y^\beta + E_{ab}(y, g)(g^{-1} \partial g)^a (g^{-1} \bar{\partial} g)^b. \end{aligned} \tag{36}$$

Note that the dependence of E_{ij} on g is fixed by condition (8). Explicitly

$$E^t(y, g) = D(g) E^t(y, e)(A(g) + B(g) E^t(y, e))^{-1}, \tag{37}$$

where e is the unit element of G, $E(y,e)$ can be chosen arbitrarily and $A(g)$ is the $(n+dimG)\times(n+dimG)$ matrix

$$A(g) \equiv \begin{pmatrix} Id & 0 \\ 0 & a(g) \end{pmatrix}, \qquad B(g) \equiv \begin{pmatrix} 0 & 0 \\ 0 & b(g) \end{pmatrix} \tag{38}$$

and $D(g)$ is given in terms of $d(g)$ in the same way as $A(g)$ in terms of $a(g)$. Needless to say, $a(g), b(g)$ and $d(g)$ are the same as in Eq. (29). As far as the dual model $\tilde{E}$ is concerned

$$\tilde{E}^t(y,\tilde{g}) = \tilde{D}(\tilde{g})\tilde{E}^t(y,e)(\tilde{A}(\tilde{g}) + \tilde{B}(\tilde{g})\tilde{E}^t(y,e))^{-1}. \tag{39}$$

Here

$$\tilde{E}^t(y,e) = (C + DE^t(y,e))(A + BE^t(y,e))^{-1}, \tag{40}$$

where

$$A = D = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = C = \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}. \tag{41}$$

5 Non-abelian modular space and conclusions

Even without a clear Abelian motivation, the natural question to ask is what is the modular space $\mathcal{M}$ of the σ-models equivalent by the $(\mathcal{G},\tilde{\mathcal{G}})$ duality. By equivalence we mean that solving one σ-model in the modular space $\mathcal{M}$, the solutions of all other models in $\mathcal{M}$ follow. It certainly does not come as a surprise that $\mathcal{M}$ is given by the structure of the Drinfeld double D. Suppose that $\mathcal{D}$ can be decomposed differently, say in $(\mathcal{K},\tilde{\mathcal{K}})$, in such a way that both algebras $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are maximally isotropic subspaces of $\mathcal{D}$ with respect to the ad-invariant bilinear form (14). A $(\mathcal{K},\tilde{\mathcal{K}})$ σ-model can be obtained by the right action of the group K on the subspace $\mathcal{E}\subset T_eD$ (see section 2). If the subspace $\mathcal{E}$ is the same as the corresponding subspace defining the $(\mathcal{G},\tilde{\mathcal{G}})$ model then two models are necessarily equivalent. Indeed, condition (17) for a surface $l(z,\bar{z})\in D$ to be obtained by lifting an extremal solution is the same whether the lifting is done from G or from K, depending just on the subspace $\mathcal{E}\in T_eD$. The explicit form of the solution $k(z,\bar{z})$ associated to a solution $g(z,\bar{z})$ is found by using the decomposition (16) from the point of view of $(\mathcal{K},\tilde{\mathcal{K}})$, i.e.

$$g(z,\bar{z})\tilde{g}(z,\bar{z}) = k(z,\bar{z})\tilde{k}(z,\bar{z}). \tag{42}$$

It seems an interesting problem to find all maximally isotropic decompositions for a generic double D. We did not attempt to do that, but we should remark that the modular space $\mathcal{M}$ is not, in general, exhausted just by the automorphisms of the double[8]. Indeed, the pure Z_2-duality $(\mathcal{G}, \tilde{\mathcal{G}}) \to (\tilde{\mathcal{G}}, \mathcal{G})$ is not an automorphism of the double if the algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ have a different structure.

Although we can relate two theories in the modular space $\mathcal{M}$ without actually solving them, it would be interesting to know whether the rich algebraic structure underlying the models can help to do that. Answering this question is somewhat indirectly related to the issues discussed in this note; nevertheless, we should probably mention how the lifting of the extremal surfaces into the double D naturally leads to a sort of the dressing transformation [23, 24, 25, 26] generating new solutions from a known one. The key point is to realize that the condition (17), when a surface $l(z, \bar{z})$ in D is the lift of some extremal solution from G (or $\tilde{G}$), is invariant with respect to the right multiplication of $l(z, \bar{z})$ by an arbitrary constant element of D. Suppose that $g(z, \bar{z})$ is an extremal surface in G. Its lift into D is given by $g(z, \bar{z})\tilde{g}(z, \bar{z})$. Now $g(z, \bar{z})\tilde{g}(z, \bar{z})g_0$ is also the lift of some extremal surface from G, i.e.

$$g(z, \bar{z})\tilde{g}(z, \bar{z})g_0 = g_1(z, \bar{z})\tilde{g}_1(z, \bar{z}), \tag{43}$$

where $g_1(z, \bar{z})$ is determined from $g(z, \bar{z}), \tilde{g}(z, \bar{z})$ and the constant element $g_0 \in G$. In other words, although the group G does not act on the target G as the isometry of the σ-model, its action on the double D via Eq. (43) does yield new solutions of the model from a known one.

So far our discussion was purely classical. It is obviously of utmost interest whether the described non-Abelian dualities relate conformal field theories (CFT) or can be even interpreted as exact symmetries of the CFT. Some interesting results were obtained in [14], where it was shown that some gauged WZNW models based on the non-semi-simple algebras are equivalent to the non-Abelian duality transformations of the WZNW actions. It is tempting to conjecture that some gauged G/H WZNW models could possess the bialgebra structure. Since extracting the classical geometry of the target is somewhat involved procedure [27] it may be difficult to see immediately whether they admit the non-commutative conservation laws. After all, a real

[8]It is so in the purely abelian case where the modular space $O(d, d, \mathbf{Z})$ is just the group of the automorphisms of the abelian double.

understanding of the sense in which the non-Abelian duality is the symmetry of the CFT requires carrying out in detail the operator mapping between a given theory and its non-Abelian dual. As a prerequisite for such investigations, we need the derivation of the $(\mathcal{G}, \tilde{\mathcal{G}})$ non-Abelian duality by a sort of path integral manipulations. Although we have made some progress in this direction, which we do not present here, this is still not sufficient to yield the complete solution of the problem. It is certainly one of the most important open issues which have to be settled in order to proceed further with the CFT application, with the problem of dilaton which we have completely ignored and with a description of the Abelian duality resembling that given by Roček and Verlinde [3]. Another interesting problem would consist in understanding the non-Abelian duality when the groups do not act freely. Let us conclude in an optimistic way: we believe that the rich algebraic structure of the presented models may turn out to be sufficient for performing a consistent quantization of the theories involved, hopefully yielding also some non-trivial CFT. It is clear that in that case we may expect new and deep applications of quantum groups in string theory.

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